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Effective balance equations for elastic composites subject to inhomogeneous potentials

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Abstract We derive the new effective governing equations for linear elastic composites subject to a body force that admits a *Helmholtz decomposition* into inhomogeneous scalar and vector potentials. We assume that the *microscale*, representing the distance between the inclusions (or fibers) in the composite, and its size (*the macroscale*) are well separated. We decouple spatial variations and assume microscale periodicity of every field. Microscale variations of the potentials induce a *locally unbounded* body force. The problem is *homogenizable*, as the results, obtained via the *asymptotic homogenization* technique, read as a well-defined linear elastic model for composites subject to a regular effective body force. The latter comprises both macroscale variations of the potentials, and non-standard contributions which are to be computed solving a well-posed elastic cell problem which is solely driven by microscale variations of the potentials. We compare our approach with an existing model for locally unbounded forces and provide a simplified formulation of the model which serves as a starting point for its numerical implementation. Our formulation is relevant to the study of active composites, such as electrosensitive and magnetosensitive elastomers.

Keywords Asymptotic homogenization · Multiscale modeling · Elastic composites · Locally unbounded force · Electroelasticity · Magnetoelasticity.

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1 Introduction

The study of real-world physical scenarios often involves the analysis of complex media possessing an internal, multiscale microstructure. Representative examples include, but are not limited to, geomaterials [60], such as rocks, sand, wood, glaciers, artificial constructs (encoding a polymer or crystalline structure) relevant to engineering applications, and biological materials, such as bones or tendons [59], organs, and (possibly malignant) cell aggregates [3, 57]. From a modeling viewpoint, it is of crucial importance to describe the macroscopic behavior of these systems, as they are typically to be investigated on a scale (*the macroscale*) dictated by their whole size, which is much larger than the one characterizing the microstructure (*the microscale*). Furthermore, it is usually challenging or almost impossible to resolve all the microstructural details at an acceptable computational cost. It is therefore natural to describe the behavior of these systems via suitable *homogenization* techniques that can lead to macroscale systems of partial differential equations (PDEs) which account for the complex interactions that take place among the different phases/constituents at the microscale level. Here we focus on elastic composites (see, e.g., [10, 20, 22, 29]) and embrace the asymptotic homogenization technique, that permits a precise prescription of the effective quantities appearing in the homogenized balance equations in terms of the microstructure. Alternative average field techniques are easier to handle when nonperiodic, nonlinear composites are considered and particularly useful when the aim only resides in the derivation of the homogenized differential model as such (see, e.g., [19] for a comparison between these two approaches).

The asymptotic homogenization technique (see, e.g., [4, 5, 18, 28, 35, 51]) provides the macroscale systems of PDEs enforcing the length scale separation between micro and macro scales (or between several scales for hierarchical materials), which are considered as independent spatial variables. Multiple scale power series expansions of the fields (in terms of the ratio between the micro and macro scales) are performed to obtain differential conditions which are used, under the assumption of microscale periodicity, to derive effective systems of PDEs for the medium on the macroscale. The microscale information is encoded in the homogenized moduli, which are to be computed solving appropriate periodic *cell problems*. The functional form and properties of the homogenized elastic problem and moduli for composites is well-known and based on the paradigmatic results concerning asymptotic homogenization (in particular, those for elliptic operators) that can be found in [35] and references therein. The authors focus on the weak formulation of several differential problems of practical interest, including the diffusion-reaction equation (also admitting locally unbounded absorption terms, which are there referred to as "rapidly oscillating potentials", not be confused with the potentials with which we are dealing in this manuscript) via both formal two-scale expansions and rigorous approaches (such as the energy method [58], translated in English in Cap. 3, [10]) to prove the convergence of the multiscale problem to the homogenized one. In [51], the authors present the generalization to the standard linear elastic problem, which is revisited via a strong form formalism in [5], where discontinuities in the elastic properties between different phases in the composite are explicitly taken into account. Asymptotic homogenization of the linear elastostatic and elastodynamic problems is carried out by the rigorous two-scale convergence approach (see [2]) in [11], where also well-posedness of the arising elastic-type cell problems is proved.

Although there exists a comprehensive literature on the subject, the focus has always been almost solely on the determination of the effective moduli, whereas possible modifications of the homogenized problem due to the properties of the external body force have not been extensively treated. Body forces are typically assumed locally homogeneous (i.e. depending on the macroscale variable only), or negligible, as done in [18, 35] and [4, 28], respectively. In [5, 11], the authors admit a periodic dependence of the body force on the microscale variable. In these cases, the given force itself (or its cell average when microscale variations are considered) plays the role of the effective body force in the homogenized balance equations. In [52], the authors assume a locally homogeneous body force in the standard derivation of the homogenized problem for elastic media characterized by locally oscillating coefficients. They further consider the particular case of an elastic medium characterized by constant elastic moduli subject to a body force which is given by an additive decomposition between a standard contribution and a locally unbounded one. The latter reads as a microscopically varying force divided by the ratio between the micro and macro scales. In this case, the authors show that the contribution related to the locally unbounded component of the body force does not appear in the homogenized balance equations. They further observe that its contribution can only be seen as a locally constant offset to the leading order asymptotic energy and when considering the local balance of the angular momentum on the periodic cell (for this reason, this contribution is referred to as a *couple-wise* force in [52]).

Here, we consider a linear elastic composite (neglecting inertia for the sake of simplicity) subject to an inhomogeneous body force that is sufficiently regular to admit a classical Helmholtz decomposition into the gradient and the curl of a scalar and a vector potential. We then apply the asymptotic homogenization technique via formal expansions and by adopting a strong form formulation approach (see [43, 44]), which has been recently enforced also to investigate various problems of practical interest such as bone modeling [46], filter efficiency [14], cardiac electrical activity [48], transport, growth, and heat exchange in tumors ([27, 40, 41, 42, 45, 47, 54]). We admit both macroscale and microscale variations of the elastic coefficients of the individual phases of the composite, as well as possible discontinuities of the elastic moduli across the interface between two different phases. We assume microscale periodicity of every field and material property, including the scalar and vector potentials introduced via the Helmholtz decomposition. As a consequence, the body force is in general locally unbounded, unless both potentials are considered homogeneous on the microscale. We derive the new effective homogenized model that describes the macroscale behavior of the composite. The novelty resides in the effective body force that appears in the homogenized balance equations. It comprises the cell average of macroscale variations of the potentials (which resembles the functional form of the classical effective body force), as well as additional contributions which involve the solution of an elastic cell problem where microscale variations of the potentials formally read as a body force. We compare our results with the "couple-wise" model given in [52]. The latter (when the standard and locally unbounded contributions to the force are microscopically and macroscopically uniform, respectively) can be recovered as particular case of our model under the same simplifying assumptions. We further discuss the particular case of multiplicative decomposition of the potentials into purely macroscopically and microscopically varying components (and under the assumption of macroscopic uniformity of the elastic moduli), which leads to a computationally feasible model, as in this case the solutions of the cell problems are the same for each macroscale point. The new model is relevant to the analysis of the deformations of linear elastic composites subject to inhomogeneous body forces, such as those arising from the application of an electric and/or magnetic field on electrosensitive and magnetosensitive continua (see, e.g., [32] for an introduction to electro- and magneto-elastic materials). A relevant example of homogenization for magnetosensitive composites is reported in [8]. In the latter work, the authors deduce the effective model in terms of the average energy via an average field approach and using appropriate variational principles. The results are derived under the assumption of rigid spherical inclusions randomly distributed in a non-magnetic linear elastic isotropic matrix and they are also specialized for the particular case of a dilute composite up to the second order of the inclusions' volume fraction.

The remainder of the work is organized as follows. In Section 2 we introduce the linear elastic problem for the composite and the Helmholtz decomposition of the body force. In Section 3 we state the spatial scale separation assumption and illustrate the asymptotic homogenization technique. In Section 4 we derive the effective governing equations for the composite. In Section 5 we discuss the results by comparing them with existing modeling approaches in section 5.1, and highlighting the applicability to electrostriction and magnetostriction in Section 5.2. In Section 6 we discuss computational aspects of the model and derive the particular case of multiplicative decomposition of the potentials. In Section 7 we present concluding remarks and further perspectives.

2 The elastic problem driven by a body force

We investigate the macroscale behavior of an elastic composite possessing oscillating and/or discontinuous coefficients and subject to an external, regular body force. In particular, any sufficiently regular vector field $\boldsymbol{f} \in \mathbb{R}^3$ admits an Helmholtz decomposition of the type

$$\boldsymbol{f}(\boldsymbol{x}) = \nabla\phi(\boldsymbol{x}) + \nabla \times \boldsymbol{A}(\boldsymbol{x}), \tag{1}$$

where ϕ and $\boldsymbol{A}$ represent inhomogeneous scalar and vector potentials, respectively. We setup the elastic problem in the material by considering the different phases explicitly according to the formulation reported in [43, 44].

We identify the composite with a bounded domain $\Omega \subset \mathbb{R}^3$, such that $\bar{\Omega} = \bar{\Omega}_I \cup \bar{\Omega}_{II}$, $\Omega_I \cap \Omega_{II} = \emptyset$, where Ω_I denotes the matrix (or host) phase and Ω_{II} a collection of N disjoint inclusions (or fibers) Ω_α defined as

$$\Omega_{II} = \bigcup_{\alpha=1}^{N} \Omega_\alpha, \;\; \alpha = 1...N. \tag{2}$$

We assume that both the matrix and the inclusions behave as linear elastic materials. The constitutive relationships for the restrictions of the stress tensor σ, denoted by σ_I and σ_α, are given by

$$\sigma^I = \mathbb{C}^I \nabla \boldsymbol{u}_I, \quad \sigma^\alpha = \mathbb{C}^{II} \nabla \boldsymbol{u}_\alpha, \tag{3}$$

where for every $\boldsymbol{x} \in \Omega$, $\boldsymbol{u}_I(\boldsymbol{x})$ and $\boldsymbol{u}_\alpha(\boldsymbol{x})$ are the restriction of the elastic displacement in Ω_I and Ω_α, respectively. The fourth rank tensors $\mathbb{C}^I(\boldsymbol{x})$, $\mathbb{C}^{II}(\boldsymbol{x})$ (with components C^I_{ijkl}, C^{II}_{ijkl} for $i,j,k,l = 1,2,3$) are the restrictions of the elasticity tensor $\mathbb{C}$ in Ω_I and in any inclusion belonging to Ω_{II}, respectively. They are equipped with major symmetry

$$C^I_{ijkl} = C^I_{klij}; \; C^{II}_{ijkl} = C^{II}_{klij} \tag{4}$$

as well as left and right minor symmetry

$$C^I_{ijkl} = C^I_{jikl}; \; C^{II}_{ijkl} = C^{II}_{jikl}; \; C^I_{ijkl} = C^I_{ijlk}; \; C^{II}_{ijkl} = C^{II}_{ijlk}, \tag{5}$$

the latter leading to

$$\mathbb{C}^I \nabla \boldsymbol{u}_I = \mathbb{C}^I \xi(\boldsymbol{u}_I); \; \mathbb{C}^{II} \nabla \boldsymbol{u}_\alpha = \mathbb{C}^{II} \xi(\boldsymbol{u}_\alpha), \tag{6}$$

where

$$\xi(\bullet) = \frac{\nabla(\bullet) + \nabla(\bullet)^{\mathsf{T}}}{2}. \tag{7}$$

The second rank tensors $\xi(\boldsymbol{u}_I)$ and $\xi(\boldsymbol{u}_\alpha)$ are the strains in the matrix and the inclusion, respectively. We enforce the stress balance equations in Ω_I and in each inclusion Ω_α. We ignore inertia and account for regular body forces according to relationship (1). We close the problem by enforcing continuity of stresses and displacements across every interface $\Gamma_\alpha := \partial\Omega_I \cap \partial\Omega_\alpha$, together with appropriate external boundary conditions on $\partial\Omega$. The resulting boundary value problem reads

$$\nabla \cdot \sigma^I + \nabla\Phi + \nabla \times \boldsymbol{A} = \boldsymbol{0} \quad \text{in } \Omega_I, \tag{8}$$
$$\nabla \cdot \sigma^\alpha + \nabla\Phi + \nabla \times \boldsymbol{A} = \boldsymbol{0} \quad \text{in } \Omega_\alpha, \tag{9}$$
$$\sigma^I \boldsymbol{n}^\alpha = \sigma^\alpha \boldsymbol{n}^\alpha \quad \text{on } \Gamma^\alpha, \tag{10}$$
$$\boldsymbol{u}_I = \boldsymbol{u}_\alpha \quad \text{on } \Gamma^\alpha, \tag{11}$$
$$+ \text{boundary conditions} \quad \text{on } \partial\Omega, \tag{12}$$

for $\alpha = 1...N$. Here, $\boldsymbol{n}^\alpha$ denotes the unit vector in $\boldsymbol{x} \in \Gamma^\alpha$ normal to the interface Γ^α pointing into the inclusion Ω_α, and the stress tensors σ^I, σ^α are given by the constitutive relationships (3). The elasticity tensors $\mathbb{C}^I$ and $\mathbb{C}^{II}$ are smooth functions of $\boldsymbol{x}$ in Ω_I and every Ω_α, respectively. The material properties can exhibit discontinuities across the interface, i.e.

$$\left(\mathbb{C}^I(\boldsymbol{x}) - \mathbb{C}^{II}(\boldsymbol{x})\right) \neq \mathsf{0} \text{ on } \Gamma^\alpha. \tag{13}$$

In the next section, we apply the asymptotic homogenization technique to derive the effective governing equations for an elastic composite subject to a body force given by the decomposition (1).

3 The asymptotic homogenization technique

We consider a typical microscale d, which characterizes the local structure and represents the distance between two adjacent inclusions, and the macroscale L, which represents the size of the whole domain Ω. We assume that these length scales are well-separated, namely:

$$\frac{d}{L} = \epsilon \ll 1. \tag{14}$$

Remark 1 (Locally unbounded body force) At this stage, we have not formally decoupled spatial variations yet. However, we are interested in *microscale* spatial heterogeneities that can also occur on the scale characterizing the distance between two inclusions, i.e. resolved on $\boldsymbol{x}/\epsilon$. As such, typical spatial variations for the scalar and vector potentials can be written as

$$\phi(\boldsymbol{x}, \boldsymbol{x}/\epsilon), \quad \boldsymbol{A}(\boldsymbol{x}, \boldsymbol{x}/\epsilon). \tag{15}$$

As a consequence, since the body force $\boldsymbol{f}(\boldsymbol{x})$ is given by the gradient and the curl of scalar and vector potentials (cf. (1)), it then exhibits a $O(1/\epsilon)$ asymptotic behavior whenever microscale variations of the type $\boldsymbol{x}/\epsilon$ are present. In this sense, our formulation considers *locally unbounded* body forces, which are regular and well-defined in the context of the elastic problem (8-12), nonetheless asymptotically behaving as $1/\epsilon$ when representation (15) holds. The systems of PDEs (8-12) turns out to be *homogenizable* assuming microscale periodicity, as done in the rest of this work. □

We enforce spatial scale decoupling exploiting condition (14) to relate d and L as follows:

$$\boldsymbol{y} := \frac{\boldsymbol{x}}{\epsilon}. \tag{16}$$

From now on, $\boldsymbol{x}$ and $\boldsymbol{y}$ denote formally independent variables, representing the macro and micro spatial coordinates, respectively. Each field and material property is assumed to be a function of both the independent variables $\boldsymbol{x}$ and $\boldsymbol{y}$. Applying the chain rule, the differential operators transform according to

$$\nabla \rightarrow \nabla_{\boldsymbol{x}} + \frac{1}{\epsilon}\nabla_{\boldsymbol{y}}. \tag{17}$$

We now assume that the elastic displacement $\boldsymbol{u}$ can be represented by multiscale expansions in powers of ϵ:

$$\boldsymbol{u}_n^{\epsilon}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{l=0}^{\infty} \boldsymbol{u}_n^{(l)}(\boldsymbol{x}, \boldsymbol{y})\epsilon^l, \; n = I, \alpha, \tag{18}$$

where we recall that relationship (18) is meant to be a *formal* representation of the displacement field, which is nonetheless acceptable under appropriate regularity assumptions, see, e.g., [11] for a discussion concerning these aspects and the relationship between the multiple scales method and the rigorous two-scale convergence approach. We further assume that every field and the elasticity tensor are $\boldsymbol{y}$-periodic. In particular, as done in [43, 44], we account for any admissible periodic cell in three dimensions by assuming that there exists a family of vectors

$$\boldsymbol{R}(\eta, \kappa, \upsilon) := \eta \boldsymbol{I}_1 + \kappa \boldsymbol{I}_2 + \upsilon \boldsymbol{I}_3, \; \eta, \kappa, \upsilon \in \mathbb{Z} \tag{19}$$

with fixed vectors $\boldsymbol{I}_1$, $\boldsymbol{I}_2$, $\boldsymbol{I}_3 \in \mathbb{R}^3$ constituting a basis of $\mathbb{R}^3$, such that, for every field and material property, collectively denoted by ψ, we have

$$\psi(\boldsymbol{x}, \boldsymbol{y}) = \psi(\boldsymbol{x}, \boldsymbol{y} + \boldsymbol{R}(\eta,\, \kappa,\, \upsilon)),\ \forall \eta,\, \kappa,\, \upsilon \in \mathbb{Z}. \tag{20}$$

This technical assumption enables us to retain microscale geometrical information by focusing on a small portion of the local structure only. We then identify the domain Ω with its corresponding periodic cell and we consider only a two-phase composite for the sake of clarity and without loss of generality (see [43] for a straightforward generalization to a multiphase problem), as shown in Figure 1. Therefore, the index α introduced in (2) is no longer necessary and we adapt our notation accordingly. From now on, the symbols Γ and $\boldsymbol{n}$ represent the interface between the matrix and the inclusion/fiber phase and the unit vector normal to the interface and pointing into the inclusion, respectively. In particular, $\boldsymbol{y}$ belongs to the cell Ω, the two individual portions of the cell are denoted by Ω_I and Ω_{II}, and the corresponding restrictions of the elastic displacement by $\boldsymbol{u}_I^\epsilon(\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{u}_{II}^\epsilon(\boldsymbol{x}, \boldsymbol{y})$, respectively.

Remark 2 (Macroscopic uniformity) The geometry of the periodic cell could also vary with respect to the macroscale spatial variable $\boldsymbol{x}$ (see e.g. [9, 18, 41, 42]), that is, one periodic cell for each macroscale point $\boldsymbol{x}$ should be considered. For the sake of simplicity, we restrict our analysis to the particular case of macroscopic uniformity, i.e. the periodic cell can be uniquely chosen independently of $\boldsymbol{x}$. In general, given a macroscopically uniform domain $\mathcal{D} = \mathcal{D}(\boldsymbol{y})$ we have, for any regular vector field $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})$

$$\nabla_{\boldsymbol{x}} \cdot \int_{\mathcal{D}(\boldsymbol{y})} \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})\ \mathrm{d}\boldsymbol{y} = \int_{\mathcal{D}(\boldsymbol{y})} \nabla_{\boldsymbol{x}} \cdot \boldsymbol{v}(\boldsymbol{x}, \boldsymbol{y})\ \mathrm{d}\boldsymbol{y}. \tag{21}$$

$\square$

We substitute the power series representation (18), together with relationship (17), into the system (8-11) and account for constitutive equations (3). As a result, multiplying each equation by a suitable power of ϵ and exploiting (6), we obtain the following multiscale system of PDEs for the elastic composite

$$\begin{aligned}
&\nabla_{\boldsymbol{y}} \cdot \Big(\mathbb{C}^I(\boldsymbol{x}, \boldsymbol{y})\xi_{\boldsymbol{y}}(\boldsymbol{u}_I^\epsilon(\boldsymbol{x}, \boldsymbol{y}))\Big) + \epsilon\nabla_{\boldsymbol{y}} \cdot \Big(\mathbb{C}^I(\boldsymbol{x}, \boldsymbol{y})\xi_{\boldsymbol{x}}(\boldsymbol{u}_I^\epsilon(\boldsymbol{x}, \boldsymbol{y}))\Big) + \\
&+ \epsilon\nabla_{\boldsymbol{x}} \cdot \Big(\mathbb{C}^I(\boldsymbol{x}, \boldsymbol{y})\xi_{\boldsymbol{y}}(\boldsymbol{u}_I^\epsilon(\boldsymbol{x}, \boldsymbol{y}))\Big) + \epsilon^2\nabla_{\boldsymbol{x}} \cdot \Big(\mathbb{C}^I(\boldsymbol{x}, \boldsymbol{y})\xi_{\boldsymbol{x}}(\boldsymbol{u}_I^\epsilon(\boldsymbol{x}, \boldsymbol{y}))\Big) \\
&+ \epsilon^2\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) + \epsilon\boldsymbol{f}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0} \quad \text{in } \Omega_I,
\end{aligned} \tag{22}$$

$$\begin{aligned}
&\nabla_{\boldsymbol{y}} \cdot \Big(\mathbb{C}^{II}(\boldsymbol{x}, \boldsymbol{y})\xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^\epsilon(\boldsymbol{x}, \boldsymbol{y}))\Big) + \epsilon\nabla_{\boldsymbol{y}} \cdot \Big(\mathbb{C}^{II}(\boldsymbol{x}, \boldsymbol{y})\xi_{\boldsymbol{x}}(\boldsymbol{u}_{II}^\epsilon(\boldsymbol{x}, \boldsymbol{y}))\Big) + \\
&+ \epsilon\nabla_{\boldsymbol{x}} \cdot \Big(\mathbb{C}^{II}(\boldsymbol{x}, \boldsymbol{y})\xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^\epsilon(\boldsymbol{x}, \boldsymbol{y}))\Big) + \epsilon^2\nabla_{\boldsymbol{x}} \cdot \Big(\mathbb{C}^{II}(\boldsymbol{x}, \boldsymbol{y})\xi_{\boldsymbol{x}}(\boldsymbol{u}_{II}^\epsilon(\boldsymbol{x}, \boldsymbol{y}))\Big) \\
&+ \epsilon^2\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{x}, \boldsymbol{y}) + \epsilon\boldsymbol{f}_{\boldsymbol{y}}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{0} \quad \text{in } \Omega_{II},
\end{aligned} \tag{23}$$

$$\mathbb{C}^I(\boldsymbol{x},\boldsymbol{y})\xi_{\boldsymbol{y}}(\boldsymbol{u}_I^\epsilon(\boldsymbol{x},\boldsymbol{y}))\boldsymbol{n} - \mathbb{C}^{II}(\boldsymbol{x},\boldsymbol{y})\xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^\epsilon(\boldsymbol{x},\boldsymbol{y}))\boldsymbol{n}$$
$$= \epsilon\mathbb{C}^{II}(\boldsymbol{x},\boldsymbol{y})\xi_{\boldsymbol{x}}(\boldsymbol{u}_{II}^\epsilon(\boldsymbol{x},\boldsymbol{y}))\boldsymbol{n} - \epsilon\mathbb{C}^{I}(\boldsymbol{x},\boldsymbol{y})\xi_{\boldsymbol{x}}(\boldsymbol{u}_{I}^\epsilon(\boldsymbol{x},\boldsymbol{y}))\boldsymbol{n} \quad \text{on } \Gamma, \tag{24}$$

$$\boldsymbol{u}_I^\epsilon(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{u}_{II}^\epsilon(\boldsymbol{x},\boldsymbol{y}) \quad \text{on } \Gamma. \tag{25}$$

The above system is defined for every macroscale point $\boldsymbol{x} \in \Omega_H$, where Ω_H denotes the macroscale domain (see Figure 1), while $\boldsymbol{y} \in \Omega$. The operators $\xi_{\boldsymbol{x}}(\bullet)$ and $\xi_{\boldsymbol{y}}(\bullet)$ represent the symmetric part of the gradient with respect to the variables $\boldsymbol{x}$ and $\boldsymbol{y}$ (cf. definition (7)), respectively, and the *macroscopic* and *local* body forces $\boldsymbol{f}_{\boldsymbol{x}}$ and $\boldsymbol{f}_{\boldsymbol{y}}$ are defined as

$$\boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{x},\boldsymbol{y}) = \nabla_{\boldsymbol{x}}\phi(\boldsymbol{x},\boldsymbol{y}) + \nabla_{\boldsymbol{x}} \times \boldsymbol{A}(\boldsymbol{x},\boldsymbol{y}) \tag{26}$$

and

$$\boldsymbol{f}_{\boldsymbol{y}}(\boldsymbol{x},\boldsymbol{y}) = \nabla_{\boldsymbol{y}}\phi(\boldsymbol{x},\boldsymbol{y}) + \nabla_{\boldsymbol{y}} \times \boldsymbol{A}(\boldsymbol{x},\boldsymbol{y}). \tag{27}$$

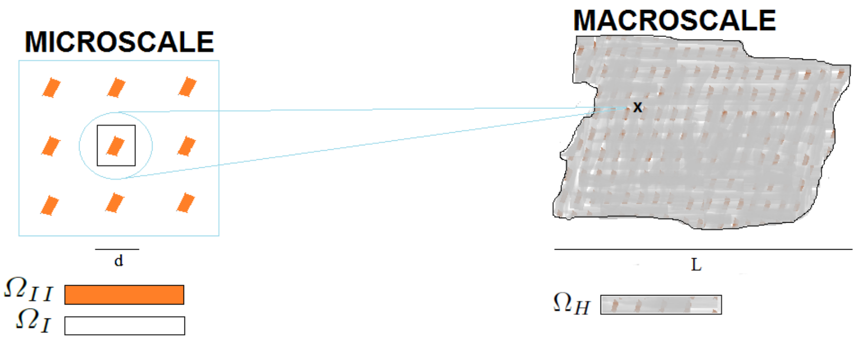

Fig. 1 A 2D schematic representing the micro and macro scales. On the right hand side, the coarse scale domain, where the fine scale structure is smoothed out, is shown. On the left hand side, a sample periodic unit representing the fine scale is encircled and the difference between the matrix and the inclusion is clearly resolved.

In the following section, we equate coefficients of ϵ^l for $l = 0, 1, ...$ in (22-25) to obtain an effective system of PDEs for the zeroth order displacement field in the homogenized domain Ω_H (see Figure 1). Since we aim at deriving a model that involves macroscale quantities only, it is convenient to define the following *cell average* operators

$$\langle\bullet\rangle = \frac{1}{|\Omega|}\int_{\Omega}\bullet\,\mathrm{d}\boldsymbol{y}; \quad \langle\bullet\rangle_I = \frac{1}{|\Omega|}\int_{\Omega_I}\bullet\,\mathrm{d}\boldsymbol{y}; \quad \langle\bullet\rangle_{II} = \frac{1}{|\Omega|}\int_{\Omega_{II}}\bullet\,\mathrm{d}\boldsymbol{y}, \tag{28}$$

where $|\Omega|$ represents the volume of the periodic cell. Next, the dependency of the fields $\boldsymbol{u}_I^{(l)}$, $\boldsymbol{u}_{II}^{(l)}$, $\mathbb{C}^I$, $\mathbb{C}^{II}$, $\boldsymbol{f}_{\boldsymbol{x}}$, $\boldsymbol{f}_{\boldsymbol{y}}$ on both $\boldsymbol{x}$ and $\boldsymbol{y}$ is meant to be understood whenever it does not appear explicitly.

4 The effective governing equations

Equating coefficients of ϵ^0 in (22-25) yields

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(0)})\right) = \boldsymbol{0} \qquad \text{in } \Omega_I, \tag{29}$$

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(0)})\right) = \boldsymbol{0} \qquad \text{in } \Omega_{II}, \tag{30}$$

$$\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(0)})\boldsymbol{n} = \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(0)})\boldsymbol{n} \qquad \text{on } \Gamma, \tag{31}$$

$$\boldsymbol{u}_I^{(0)} = \boldsymbol{u}_{II}^{(0)} \qquad \text{on } \Gamma, \tag{32}$$

whereas equating coefficients of ϵ^1 in (22-25) leads to

$$\begin{aligned} &\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(1)})\right) + \nabla_{\boldsymbol{x}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(0)})\right) \\ &+\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{x}}(\boldsymbol{u}_I^{(0)})\right) + \boldsymbol{f}_{\boldsymbol{y}} = \boldsymbol{0} \quad \text{in } \Omega_I, \end{aligned} \tag{33}$$

$$\begin{aligned} &\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(1)})\right) + \nabla_{\boldsymbol{x}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(0)})\right) \\ &+\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{x}}(\boldsymbol{u}_{II}^{(0)})\right) + \boldsymbol{f}_{\boldsymbol{y}} = \boldsymbol{0} \quad \text{in } \Omega_{II}, \end{aligned} \tag{34}$$

$$\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(1)})\boldsymbol{n} - \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(1)})\boldsymbol{n} = \mathbb{C}^{II} \xi_{\boldsymbol{x}}(\boldsymbol{u}_{II}^{(0)})\boldsymbol{n} - \mathbb{C}^I \xi_{\boldsymbol{x}}(\boldsymbol{u}_I^{(0)})\boldsymbol{n} \quad \text{on } \Gamma, \tag{35}$$

$$\boldsymbol{u}_{II}^{(1)} = \boldsymbol{u}_{II}^{(1)} \quad \text{on } \Gamma. \tag{36}$$

Equating coefficients of ϵ^2 in (22-24) we obtain

$$\begin{aligned} &\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(2)})\right) + \nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{x}}(\boldsymbol{u}_I^{(1)})\right) + \\ &+\nabla_{\boldsymbol{x}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(1)})\right) + \nabla_{\boldsymbol{x}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{x}}(\boldsymbol{u}_I^{(0)})\right) + \boldsymbol{f}_{\boldsymbol{x}} = \boldsymbol{0} \quad \text{in } \Omega_I, \end{aligned} \tag{37}$$

$$\begin{aligned} &\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(2)})\right) + \nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{x}}(\boldsymbol{u}_{II}^{(1)})\right) + \\ &+\nabla_{\boldsymbol{x}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(1)})\right) + \nabla_{\boldsymbol{x}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{x}}(\boldsymbol{u}_{II}^{(0)})\right) + \boldsymbol{f}_{\boldsymbol{x}} = \boldsymbol{0} \quad \text{in } \Omega_{II}, \end{aligned} \tag{38}$$

$$\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(2)})\boldsymbol{n} - \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(2)})\boldsymbol{n} = \mathbb{C}^{II} \xi_{\boldsymbol{x}}(\boldsymbol{u}_{II}^{(1)})\boldsymbol{n} - \mathbb{C}^I \xi_{\boldsymbol{x}}(\boldsymbol{u}_I^{(1)})\boldsymbol{n} \quad \text{on } \Gamma. \tag{39}$$

The solutions of the periodic cell problem (29-32) are $\boldsymbol{y}$-constant functions. Hence, since continuity across the interfaces Γ holds, the leading order displacement field reads

$$\boldsymbol{u}^{(0)}(\boldsymbol{x}) = \boldsymbol{u}_I^{(0)}(\boldsymbol{x}) = \boldsymbol{u}_{II}^{(0)}(\boldsymbol{x}) \tag{40}$$

and we also simplify the notation defining

$$\bar{\boldsymbol{u}}(\boldsymbol{x}) = \boldsymbol{u}^{(0)}(\boldsymbol{x}). \tag{41}$$

Employing relationships (40-41) in equations (33-36) and accounting for (40), we obtain the following differential problem for $\boldsymbol{u}^{(1)}$, which reads, in terms of the restrictions $\boldsymbol{u}_I^{(1)}(\boldsymbol{x},\boldsymbol{y})$ and $\boldsymbol{u}_{II}^{(1)}(\boldsymbol{x},\boldsymbol{y})$

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(1)})\right) + \nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})\right) + \boldsymbol{f}_{\boldsymbol{y}} = \boldsymbol{0} \quad \text{in } \Omega_I, \tag{42}$$

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(1)})\right) + \nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})\right) + \boldsymbol{f}_{\boldsymbol{y}} = \boldsymbol{0} \quad \text{in } \Omega_{II}, \tag{43}$$

$$\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(1)})\boldsymbol{n} - \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(1)})\boldsymbol{n} = \left(\mathbb{C}^{II} - \mathbb{C}^I\right) \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})\boldsymbol{n} \quad \text{on } \Gamma, \tag{44}$$

$$\boldsymbol{u}_I^{(1)} = \boldsymbol{u}_{II}^{(1)} \quad \text{on } \Gamma. \tag{45}$$

The problem (42-45) is a linear, elastic-type periodic boundary value problem equipped with continuity and stress jump interface conditions on Γ.

Remark 3 (Compatibility condition of the cell problems) The cell problem (42-45) is a linear elastic-type cell problem, equipped with periodic conditions on $\partial\Omega$ and stress-jump interface conditions on Γ. Its classical counterpart (see, e.g., [5, 43, 44]), which corresponds to the particular case $\boldsymbol{f}_{\boldsymbol{y}} = \boldsymbol{0}$, admits a unique solution up to a $\boldsymbol{y}$-constant function, as shown for example in [11] in a weak-form rigorous setting which also accounts for jump discontinuities of the elastic moduli. In our case, the compatibility condition reads

$$\begin{aligned}&\int_{\Omega_I} \nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(1)})\right) \mathrm{d}\boldsymbol{y} + \int_{\Omega_{II}} \nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(1)})\right) \mathrm{d}\boldsymbol{y} = \\ &\int_\Gamma \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(1)})\boldsymbol{n} - \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(1)})\boldsymbol{n}\right) \mathrm{dS} = \int_\Gamma (\mathbb{C}^{II} - \mathbb{C}^I)\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})\boldsymbol{n}\, \mathrm{dS},\end{aligned} \tag{46}$$

where, on the left hand side, we have applied the divergence[1] theorem with respect to $\boldsymbol{y}$ and considered that the contributions on the periodic cell boundary $\partial\Omega$ cancel, and on the right hand side we have enforced interface condition (44). The left hand side of compatibility condition (46) can be rewritten exploiting relationships (42-43) as

$$-\int_{\Omega_I} \nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})\right) \mathrm{d}\boldsymbol{y} - \int_{\Omega_{II}} \nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})\right) \mathrm{d}\boldsymbol{y} - \int_\Omega \boldsymbol{f}_{\boldsymbol{y}}\, \mathrm{d}\boldsymbol{y}, \tag{47}$$

where we recall that the regular vector field $\boldsymbol{f}_{\boldsymbol{y}}$ does not exhibit jump discontinuities and is therefore well defined on the whole periodic cell Ω. We further apply the divergence theorem with respect to $\boldsymbol{y}$ to the left hand side of equation (47) and we substitute it in the condition (46), which rewrites

$$\int_\Gamma (\mathbb{C}^{II} - \mathbb{C}^I)\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})\boldsymbol{n}\, \mathrm{dS} - \int_\Omega \boldsymbol{f}_{\boldsymbol{y}}\, \mathrm{d}\boldsymbol{y} = \int_\Gamma (\mathbb{C}^{II} - \mathbb{C}^I)\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})\boldsymbol{n}\, \mathrm{dS}. \tag{48}$$

We then deduce that the compatibility condition for the cell problem (42-45) is satisfied if and only if

$$\int_\Omega \boldsymbol{f}_{\boldsymbol{y}}\, \mathrm{d}\boldsymbol{y} = \boldsymbol{0}. \tag{49}$$

[1] As $\boldsymbol{n}$ is pointing from the matrix into the inclusion/fiber, the unit normal vector pointing from the inclusion into the matrix is given by $-\boldsymbol{n}$.

The volume integral of a generic periodic function is in general nonzero. However, substituting the definition of the local body force (27) into (49) we obtain

$$\int_{\Omega} \boldsymbol{f_y}\, \mathrm{d}\boldsymbol{y} = \int_{\Omega} \nabla_{\boldsymbol{y}}\phi + \nabla_{\boldsymbol{y}} \times \boldsymbol{A}\, \mathrm{d}\boldsymbol{y} = \int_{\partial\Omega} \phi \boldsymbol{n}_c \,\mathrm{dS} + \int_{\partial\Omega} \boldsymbol{n}_c \times \boldsymbol{A}\,\mathrm{dS}, \tag{50}$$

where $\boldsymbol{n}_c$ is the outward unit vector normal to the cell boundary $\partial\Omega$. Since the potentials ϕ and $\boldsymbol{A}$ are $\boldsymbol{y}$-periodic, the right hand side of (50) is identically zero, i.e. (49) and therefore the compatibility condition (46) is satisfied. In particular, both terms (involving the scalar and vector potentials) reduce individually to zero. □

The differential problem (42-45) therefore admits a unique solution up to a $\boldsymbol{y}$-constant function, as its classical counterpart. Exploiting linearity, the restrictions $\boldsymbol{u}_I^{(1)}$, $\boldsymbol{u}_{II}^{(1)}$ of the solution $\boldsymbol{u}^{(1)}$ are given by the ansätze

$$\begin{aligned}
\boldsymbol{u}_I^{(1)}(\boldsymbol{x},\boldsymbol{y}) &= \chi^I(\boldsymbol{x},\boldsymbol{y})\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}(\boldsymbol{x})) + \tilde{\boldsymbol{u}}_I(\boldsymbol{x},\boldsymbol{y}),\\
\boldsymbol{u}_{II}^{(1)}(\boldsymbol{x},\boldsymbol{y}) &= \chi^{II}(\boldsymbol{x},\boldsymbol{y})\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}(\boldsymbol{x})) + \tilde{\boldsymbol{u}}_{II}(\boldsymbol{x},\boldsymbol{y}).
\end{aligned} \tag{51}$$

The third rank tensor χ, with restrictions in Ω_I and Ω_{II} given by χ^I and χ^{II}, respectively, is the solution of the following periodic cell problems:

$$\frac{\partial}{\partial y_j}\left(C^I_{ijpq}\xi^{kl}_{pq}(\chi^I)\right) = -\frac{\partial C^I_{ijkl}}{\partial y_j} \text{ in } \Omega_I, \tag{52}$$

$$\frac{\partial}{\partial y_j}\left(C^{II}_{ijpq}\xi^{kl}_{pq}(\chi^{II})\right) = -\frac{\partial C^{II}_{ijkl}}{\partial y_j} \text{ in } \Omega_{II}, \tag{53}$$

$$C^I_{ijpq}\xi^{kl}_{pq}(\chi^I)n_j - C^{II}_{ijpq}\xi^{kl}_{pq}(\chi^{II})n_j = (C^{II} - C^I)_{ijkl}n_j \text{ on } \Gamma, \tag{54}$$

$$\chi^I_{ikl} = \chi^{II}_{ikl} \text{ on } \Gamma, \tag{55}$$

where we set

$$\xi^{kl}_{pq}(\chi^I) = \frac{1}{2}\left(\frac{\partial \chi^I_{pkl}}{\partial y_q} + \frac{\partial \chi^I_{qkl}}{\partial y_p}\right);\ \xi^{kl}_{pq}(\chi^{II}) = \frac{1}{2}\left(\frac{\partial \chi^{II}_{pkl}}{\partial y_q} + \frac{\partial \chi^{II}_{qkl}}{\partial y_p}\right), \tag{56}$$

and sum over repeated indices p, q, j is understood. We further define the auxiliary fourth rank tensor $\mathbb{M}(\boldsymbol{x},\boldsymbol{y})$, with restrictions $\mathbb{M}^I(\boldsymbol{x},\boldsymbol{y})$ and $\mathbb{M}^{II}(\boldsymbol{x},\boldsymbol{y})$, defined componentwise as

$$\begin{aligned}
M^I_{pqkl} &= \xi^{kl}_{pq}(\chi^I) = \frac{1}{2}\left(\frac{\partial \chi^I_{pkl}}{\partial y_q} + \frac{\partial \chi^I_{qkl}}{\partial y_p}\right),\\
M^{II}_{pqkl} &= \xi^{kl}_{pq}(\chi^{II}) = \frac{1}{2}\left(\frac{\partial \chi^{II}_{pkl}}{\partial y_q} + \frac{\partial \chi^{II}_{qkl}}{\partial y_p}\right),
\end{aligned} \tag{57}$$

where we recall that the above tensors are equipped with both minor symmetries (but in general not major symmetry) and zero average on the periodic cell (see Theorem 1, page 195 [44]). The problem (52-55) is then closed by periodic conditions on $\partial\Omega$. We recall that differential problems of this kind are well-posed up to a constant, which can be uniquely determined imposing a further condition. A common choice is to impose that the cell average of the auxiliary variable is zero, as done for example in [11], or fix the solution in one point of the cell, as in [43]. The vector field $\tilde{\boldsymbol{u}}$, with restrictions $\tilde{\boldsymbol{u}}_I$ and $\tilde{\boldsymbol{u}}_{II}$, is the solution of the following elastic periodic cell problem:

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I)\right) + \boldsymbol{f}_{\boldsymbol{y}} = \boldsymbol{0} \quad \text{in } \Omega_I, \tag{58}$$

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II})\right) + \boldsymbol{f}_{\boldsymbol{y}} = \boldsymbol{0} \quad \text{in } \Omega_{II}, \tag{59}$$

$$\mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I)\boldsymbol{n} - \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II})\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{60}$$

$$\tilde{\boldsymbol{u}}_I = \tilde{\boldsymbol{u}}_{II} \quad \text{on } \Gamma. \tag{61}$$

Once a further condition is imposed to achieve uniqueness, the above cell problem is well-posed by means of the classical existence and uniqueness results (see, e.g., [11]) and recalling that the compatibility condition is satisfied as (49) holds.

Remark 4 (Properties of the cell problems) Nontrivial solutions of the cell problems (52-55) (which correspond to six elastic-type cell problems for every fixed (k,l), $k,l = 1,2,3$, $k \geq l$, as $\mathbb{M}$ is minor symmetric, see [43, 44]) are driven by local variations of the elastic constants within the composite, which formally appear as body forces on the right hand side of (52-53), as well as the interface loads in the stress discontinuity conditions (54), see also [43, 44]. In the latter case, the contribution is directly related to the difference in the elastic constants between the matrix and the inclusion/fiber, and to the interface geometry. The cell problem (58-61) formally reads as an elastic periodic cell problem solely driven by the local body force $\boldsymbol{f}_{\boldsymbol{y}}$ and equipped with continuity of stresses and displacements across the inteface Γ. □

We now aim to formulate the effective governing equations for the composite. We apply the integral average operators (28) over Ω_I and Ω_{II} in equation (37) and equation (38), respectively. We then sum every resulting contribution and apply the divergence theorem in $\boldsymbol{y}$, such that, rearranging terms, we obtain:

$$\begin{aligned} &\frac{1}{|\Omega|}\left[\int_\Gamma \mathbb{C}^I \xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(2)})\boldsymbol{n}\ \mathrm{dS} - \int_\Gamma \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(2)})\boldsymbol{n}\ \mathrm{dS} + \right. \\ &\left. + \int_\Gamma \mathbb{C}^I \xi_{\boldsymbol{x}}(\boldsymbol{u}_I^{(1)})\boldsymbol{n}\ \mathrm{dS} - \int_\Gamma \mathbb{C}^{II} \xi_{\boldsymbol{x}}(\boldsymbol{u}_{II}^{(1)})\boldsymbol{n}\ \mathrm{dS}\right] + \\ &+ \frac{1}{|\Omega|}\int_{\Omega_I} \nabla_{\boldsymbol{x}} \cdot \left(\mathbb{C}^I(\xi_{\boldsymbol{y}}(\boldsymbol{u}_I^{(1)}) + \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}))\right)\ \mathrm{d}\boldsymbol{y} + \\ &+ \frac{1}{|\Omega|}\left[\int_{\Omega_{II}} \nabla_{\boldsymbol{x}} \cdot \left(\mathbb{C}^{II}(\xi_{\boldsymbol{y}}(\boldsymbol{u}_{II}^{(1)}) + \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}))\right)\ \mathrm{d}\boldsymbol{y}\right] + \langle \boldsymbol{f}_{\boldsymbol{x}} \rangle = \boldsymbol{0}, \end{aligned} \tag{62}$$

where the contributions over the cell boundary $\partial\Omega$ cancel due to $\boldsymbol{y}$-periodicity. We account for relationship (39), such that also the contributions over the interface Γ in (62) cancel. Finally, we enforce ansätze (51) and macroscopic uniformity (21) to deduce the following effective governing equations for every $\boldsymbol{x} \in \Omega_H$, namely

$$\nabla_{\boldsymbol{x}} \cdot \left(\tilde{\mathbb{C}}(\boldsymbol{x})\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}(\boldsymbol{x}))\right) + \hat{\boldsymbol{f}}(\boldsymbol{x}) = \boldsymbol{0}. \tag{63}$$

The effective elasticity tensor $\tilde{\mathbb{C}}$ is given by:

$$\tilde{\mathbb{C}} = \left\langle \mathbb{C}^I + \mathbb{C}^I \mathbb{M}^I \right\rangle_I + \left\langle \mathbb{C}^{II} + \mathbb{C}^{II} \mathbb{M}^{II} \right\rangle_{II} \tag{64}$$

or, componentwise:

$$\tilde{C}_{ijkl} = \left\langle C^{I}_{ijkl} + C^{I}_{ijpq} M^{I}_{pqkl} \right\rangle_{I} + \left\langle C^{II}_{ijkl} + C^{II}_{ijpq} M^{II}_{pqkl} \right\rangle_{II}, \tag{65}$$

and the effective body force reads:

$$\begin{aligned}\hat{\boldsymbol{f}} &= \langle \boldsymbol{f_x} \rangle + \nabla_{\boldsymbol{x}} \cdot \left\langle \mathbb{C}^{I} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}^{I}) \right\rangle_{I} + \nabla_{\boldsymbol{x}} \cdot \left\langle \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}^{II}) \right\rangle_{II} \\ &= \nabla_{\boldsymbol{x}} \langle \phi \rangle + \nabla_{\boldsymbol{x}} \times \langle \boldsymbol{A} \rangle + \nabla_{\boldsymbol{x}} \cdot \left\langle \mathbb{C}^{I} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}^{I}) \right\rangle_{I} + \nabla_{\boldsymbol{x}} \cdot \left\langle \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}^{II}) \right\rangle_{II}. \end{aligned} \tag{66}$$

The homogenized balance equations (63) can also be formulated in terms of the average of the leading order stress $\sigma^{(0)}$. Its restrictions to Ω_I and Ω_{II} are obtained applying the asymptotic homogenization steps (17-18) to the constitutive relationships (3):

$$\sigma^{(0)I} = \mathbb{C}^{I} \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) + \mathbb{C}^{I} \xi_{\boldsymbol{y}}(\boldsymbol{u}^{(1)}_{I}), \quad \sigma^{(0)II} = \mathbb{C}^{II} \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) + \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\boldsymbol{u}^{(1)}_{II}). \tag{67}$$

We then enforce the ansätze (51) and definitions (57) to deduce

$$\begin{aligned}\sigma^{(0)I} &= (\mathbb{C}^{I} + \mathbb{C}^{I}\mathbb{M}^{I}) \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) + \mathbb{C}^{I} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}^{I}), \\ \sigma^{(0)II} &= (\mathbb{C}^{II} + \mathbb{C}^{II}\mathbb{M}^{II}) \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) + \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}^{II}). \end{aligned} \tag{68}$$

The relationships above can be exploited to equivalently formulate the effective balance equation (63) in terms of $\sigma^{(0)}$ as follows

$$\nabla_{\boldsymbol{x}} \cdot \left(\left\langle \sigma^{(0)I} \right\rangle_{I} + \left\langle \sigma^{(0)II} \right\rangle_{II} \right) + \langle \boldsymbol{f_x} \rangle = \boldsymbol{0} \tag{69}$$

or

$$\nabla_{\boldsymbol{x}} \cdot (\sigma_E + \sigma_L) + \langle \boldsymbol{f_x} \rangle = \boldsymbol{0}, \tag{70}$$

where σ_E is the standard effective stress tensor for linear elastic composites, while the auxiliary tensor σ_L formally reads as the average of the auxiliary stress which is involved in the elastic cell problem (58-61), namely

$$\sigma_E = \tilde{\mathbb{C}} \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}), \quad \sigma_L = \left\langle \mathbb{C}^{I} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{I}) \right\rangle_{I} + \left\langle \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}) \right\rangle_{II}. \tag{71}$$

In the next section we discuss and interpret the results, and provide a comparison with the existing modeling approach for locally unbounded forces reported in [52].

5 Discussion of the results

We have derived the new differential model (63) that describes the macroscale behavior of linear elastic composites subject to possibly inhomogeneous and locally unbounded (see remark 1) body forces that admit the standard Helmholtz decomposition (1). The novelty resides in the effective volume load $\hat{\boldsymbol{f}}(\boldsymbol{x})$ given by relationship (66), which encodes both macroscale and microscale variations of the given potentials. The former contribution resembles the same functional form that would appear in the standard formulation, as it reads as the cell average of the macroscopic body force $\boldsymbol{f_x}$ (cf. (26)); the latter requires the solution of the arising elastic cell problem (58-61), which is solely driven by microscale variations of the scalar and vector potentials, namely $\boldsymbol{f}_{\boldsymbol{y}}$ (cf. (27)).

Our result reduces to the standard case, where only the body force itself or its cell average appears in the homogenized balance equations, when microscale variations of the potentials can be neglected. In this case $\boldsymbol{f}_y = 0$, and $\boldsymbol{f} = \boldsymbol{f}_{\boldsymbol{x}}(\boldsymbol{x})$ no longer depends on the microscale. The solution of the problem (58-61) is a constant, and therefore every term involving microscale gradients of the cell problem's solution in (66) vanishes. As a result, the effective body force (66) is simply given by $\boldsymbol{f}$.

We show that the model (8-12) is *homogenizable*, even though the body force acting on the composite is in general locally unbounded (see Remark 1). The balance equations (63) obtained via asymptotic homogenization of the problem (8-12) read as a well-defined linear elastic model (once appropriate external boundary conditions on $\partial\Omega_H$ are prescribed) in terms the leading order displacement $\bar{\boldsymbol{u}}$ in the homogenized domain Ω_H. The fourth rank operator $\tilde{\mathbb{C}}$ is standard and possesses every property that characterizes the effective elasticity tensor for composites, i.e. major and minor symmetries, positive definiteness, and the Voigt and Reuss energetic bounds (see [11] and Theorem 2, page 198, [44]). The vector field $\hat{\boldsymbol{f}}$ given by (66) plays the role of an effective body force and is a regular function that depends on the macroscale only and is to be computed enforcing the solution of the well-defined (see Remark 3) cell problem (58-61).

As we have remarked in the Introduction, the implications of peculiar external forces acting on elastic composites (and in general, appearing in classical multiscale problems arising in mathematical-physics) have not been extensively investigated. However, in [52] (pages 94-96), the authors account for the role of locally unbounded body forces under a number of simplifying assumptions, and the structure of their formulation can be viewed as a particular case of our model, as depicted below.

5.1 Comparison with the previous modeling approach [52]

In [52] the authors investigate the role of a locally unbounded force of the type

$$\tilde{\boldsymbol{f}}(\boldsymbol{x},\boldsymbol{y}) + \varphi(x)\frac{1}{\epsilon}\boldsymbol{F}(\boldsymbol{y}), \tag{72}$$

where φ is a locally uniform scalar field and $\tilde{\boldsymbol{f}}$, $\boldsymbol{F}$ are $\boldsymbol{y}$-periodic vectors. The authors focus on the action of the body force (72) on a linear elastic material characterized by globally uniform stiffness tensor, that is, in our notation[2]

$$\mathbb{C} = \mathbb{C}^{I} = \mathbb{C}^{II}, \tag{73}$$

where $\mathbb{C}$ is the constant elasticity tensor. The authors show that, under the assumptions (72-73), asymptotic homogenization of the elastic problem leads to homogenized balance equations that do not depend on the locally unbounded component

[2] In [52], pages 94-96, the authors enforce a weak formulation of the problem and every vector and tensor quantity is denoted componentwise. The elasticity tensor $\mathbb{C}$, strain operator $\xi(\bullet)$, average operator $\langle\bullet\rangle$, the force $\tilde{\boldsymbol{f}}$, the function φ and the periodic cell Ω are there denoted by a, e, $\tilde{(\bullet)}$, f, ϕ, and Y, respectively.

of the force (i.e. on $\boldsymbol{F}$ and φ) and simply read

$$\nabla_{\boldsymbol{x}} \cdot \left\langle \sigma^{(0)} \right\rangle + \left\langle \tilde{\boldsymbol{f}} \right\rangle = \boldsymbol{0}, \quad \left\langle \sigma^{(0)} \right\rangle = \mathbb{C}\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}). \tag{74}$$

They further note that the average zero-th order asymptotic energy, defined as the leading order term of the functional

$$W^{\epsilon} = \langle \sigma^{\epsilon} : \xi(\boldsymbol{u}^{\epsilon}) \rangle \,, \tag{75}$$

can be written as the sum of the macroscale elastic energy and a locally constant contribution, namely

$$W^{(0)} = \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}):(\mathbb{C}\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})) + \varphi(\boldsymbol{x})\beta, \tag{76}$$

where the operation : is defined as the standard double contraction between two second rank tensors A and B, componentwise

$$\mathsf{A}{:}\mathsf{B} = A_{ij}B_{ij}. \tag{77}$$

The constant β in (76) reads

$$\beta = \langle \xi_{\boldsymbol{y}}(\boldsymbol{w}) : \mathbb{C}\xi_{\boldsymbol{y}}(\boldsymbol{w}) \rangle \tag{78}$$

where $\boldsymbol{w}$ solves a cell problem analogous to (58-61), for a single phase elastic material with $\boldsymbol{f}_{\boldsymbol{y}}$ replaced by $\boldsymbol{F}$. Finally, they remark that according to their results, the leading order stress $\sigma^{(0)}$ satisfies the microscale balance of angular momentum with respect to the component of the locally unbounded force, i.e.

$$\int_{\partial\Omega} \boldsymbol{y} \times \sigma^{(0)}\boldsymbol{n}_c \, \mathrm{dS} = \int_{\Omega} \boldsymbol{y} \times \boldsymbol{F} \, \mathrm{d}\boldsymbol{y}. \tag{79}$$

Here we are tackling a different problem, as we consider a Helmholtz decomposition of the external body force and inhomogeneous (and potentially discontinuous) elastic coefficients, which typically characterize composite materials. As such, locally unboundedness is not imposed via an artificial scaling, as it naturally arises accounting for microscale variations of the potentials. However, the problem discussed in [52], when assuming a pure additive decomposition of the body force (72) into a purely microscale and macroscale contribution, that is

$$\varphi = 1, \quad \tilde{\boldsymbol{f}} = \tilde{\boldsymbol{f}}(\boldsymbol{x}), \tag{80}$$

can be considered a particular case of our formulation. We recover the structure of all the results (74), (76), and (79) identifying their $\tilde{\boldsymbol{f}}$ and $\boldsymbol{F}$ with our macroscopic and local forces $\boldsymbol{f}_{\boldsymbol{x}}$ and $\boldsymbol{f}_{\boldsymbol{y}}$, whenever also the latter only depend on the macroscale and microscale, respectively. This particular case is achieved considering an additive decomposition of the potentials into purely macroscopically and microscopically varying components, cf. (26-27).

In fact, the cell problem (52-55) is a redundant identity for a globally uniform $\mathbb{C}$, as neither variations nor discontinuities of the elastic constants are present. The solution is a constant and the auxiliary tensor

$$\mathbb{M} = 0. \tag{81}$$

The cell problem for the auxiliary variable $\tilde{\boldsymbol{u}}$ simplifies to

$$\nabla_{\boldsymbol{y}} \cdot (\mathbb{C}\xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}})) + \boldsymbol{f}_{\boldsymbol{y}} = \boldsymbol{0} \quad \text{in } \Omega, \tag{82}$$

with periodic conditions on $\partial\Omega$ and equipped with a further uniqueness condition, for example $\langle \tilde{\boldsymbol{u}} \rangle = 0$. The cell problem (82) corresponds to the one solved by the auxiliary variable $\boldsymbol{w}$ in [52] when identifying $\boldsymbol{f}_{\boldsymbol{y}}$ with the local component of the force $\boldsymbol{F}$ given in [52]. In our case, we do not need to assume the compatibility condition (49) (as done in [52] for $\boldsymbol{F}$), as this is automatically satisfied by means of its definition in terms of the local Helmholtz decomposition (27), see Remark 3. Enforcing (73) and (81), the effective elasticity tensor (64) and body force (66) read

$$\tilde{\mathbb{C}} = \mathbb{C}, \quad \hat{\boldsymbol{f}} = \boldsymbol{f}_{\boldsymbol{x}} + \nabla_{\boldsymbol{x}} \cdot \langle \mathbb{C}\xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}) \rangle \,. \tag{83}$$

However, the term involving the solution of the cell problem on the right hand side of (83) can be rewritten, for a constant elasticity tensor, as

$$\langle \mathbb{C}\xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}) \rangle = \mathbb{C} \langle \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}) \rangle = \frac{1}{2}\mathbb{C} \left(\int_{\partial\Omega} \tilde{\boldsymbol{u}} \otimes \boldsymbol{n}_c + \boldsymbol{n}_c \otimes \tilde{\boldsymbol{u}} \right) = 0, \tag{84}$$

where the last two terms on the boundary $\partial\Omega$ reduce individually to zero due to $\boldsymbol{y}$-periodicity. Hence, the homogenized balance equations reduce to

$$\nabla_{\boldsymbol{x}} \cdot (\mathbb{C}\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})) + \boldsymbol{f}_{\boldsymbol{x}} = \boldsymbol{0}, \tag{85}$$

which corresponds to (74) when identifying $\boldsymbol{f}_{\boldsymbol{x}}$ with $\tilde{\boldsymbol{f}}$. In this particular case, our results agree with those reported in [52] and the locally unbounded contribution to the force plays no role in determining the effective macroscale model. In order to emphasize the observations reported in (79), we first notice that the problem for $\boldsymbol{u}^{(1)}$ (42-45) can be in general written in terms of the leading order stress $\sigma^{(0)}$ (cf. (67)). In the particular case of a globally constant $\mathbb{C}$ it reads

$$\nabla_{\boldsymbol{y}} \cdot \sigma^{(0)} + \boldsymbol{f}_{\boldsymbol{y}} = \boldsymbol{0} \quad \text{in } \Omega, \tag{86}$$

where

$$\sigma^{(0)} = \mathbb{C}\xi_{\boldsymbol{y}}(\boldsymbol{u}^{(1)}) + \mathbb{C}\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}). \tag{87}$$

Since the problem (86) reads formally as a standard balance of linear momentum over the periodic cell Ω, it then implies balance of angular momentum (79) as $\sigma^{(0)}$ is symmetric due to the left minor symmetry of $\mathbb{C}$. Finally, the zeroth order term of the asymptotic energy (75) reads, in general

$$W^{(0)} = \left\langle \sigma^{(0)} : (\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) + \xi_{\boldsymbol{y}}(\boldsymbol{u}^{(1)})) \right\rangle = \tag{88}$$

$$\langle ((\mathbb{C} + \mathbb{C}\mathbb{M})\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) + \mathbb{C}\xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}})) : ((\mathbb{I} + \mathbb{M})\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) + \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}})) \rangle, \tag{89}$$

where we have exploited the asymptotic homogenization steps (17-18) in definitions (3), (7), have considered that $\bar{\boldsymbol{u}}$ does not depend on the microscale (cf. (40-41)), and expressed $\xi_{\boldsymbol{y}}(\boldsymbol{u}^{(1)})$ in terms of ansätze (51) together with definitions (57). We now account for a globally constant $\mathbb{C}$, which implies (81), to obtain

$$W^{(0)} = \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) : \mathbb{C}\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) + \langle \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}) : \mathbb{C}\xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}) \rangle + \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) : \mathbb{C} \langle \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}) \rangle + \langle \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}) \rangle : \mathbb{C}\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}). \tag{90}$$

Since $\langle \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}})\rangle = 0$ due to $\boldsymbol{y}$-periodicity (cf. (84)), we finally obtain

$$W^{(0)} = \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}){:}\mathbb{C}\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) + \langle \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}){:}\mathbb{C}\xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}})\rangle\,, \tag{91}$$

which coincides with the result (76) reported in [52] accounting for definition (78), assumption (80), and identifying our $\tilde{\boldsymbol{u}}$ with the auxiliary vector $\boldsymbol{w}$ defined in [52] (cf. cell problem (82)).

The comparison between our model and that reported in [52] further elucidates the fact that microscale variations of the potentials induce a locally unbounded contribution to the body force, see also Remark 1. The major feature of our model resides in the form of the effective body force (66), which encodes also microscale variations of the potentials. In our case, the solution of the auxiliary cell problem (52-55) is therefore necessary to solve the macroscale problem. In [52], as a direct consequence of global uniformity of the elastic coefficients, this is not the case as it only appears in the local balance of linear momentum (86), which in turn implies the angular momentum balance on the periodic cell (79), and the constant offset to the asymptotic energy (91). The latter peculiarities are also present in our model. In particular, the problem (42-45) in our general setting can likewise be written in terms of the leading order stress tensor given by (67). This way, it formally reads as local balance of linear momentum on the periodic cell which still implies the corresponding balance of angular momentum as a direct consequence of the symmetry of $\sigma^{(0)}$ and continuity of tractions

$$\sigma^{(0)I}\boldsymbol{n} = \sigma^{(0)II}\boldsymbol{n} \quad \text{on } \mathit{\Gamma}, \tag{92}$$

which is obtained rewriting (35) in terms of $\sigma^{(0)}$. In both cases, we could interpret $\sigma^{(0)}$ as the stress tensor satisfying the local equilibrium equations on the periodic cell in the presence of the component of the force related to local unboundedness (that is $\boldsymbol{f_y}$ in this work, and $\boldsymbol{F}$ in [52]). The average of $\sigma^{(0)}$, which in our case comprises the additional contribution encoded in the auxiliary tensor σ_L (cf. equation (71)) equilibrates instead the cell average of the macroscopic volume load (i.e $\boldsymbol{f_x}$ here and $\tilde{\boldsymbol{f}}$ in [52]). Finally, the form of the functional $W^{(0)}$ defined in equation (89) differs, in general, from that found in [52], as in our case the coefficients are not globally constant and $\mathbb{M} \neq 0$. However, the physically meaningful macroscale energy density is in any case given by the quadratic form

$$W(\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})) = \frac{1}{2}\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) : \tilde{\mathbb{C}}\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) \tag{93}$$

which satisfies

$$\frac{\partial W}{\partial \xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})} = \tilde{\mathbb{C}}\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}}) = \sigma_E, \tag{94}$$

whereas contributions related to microscale variations of the potentials affect the average local auxiliary stress σ_L only (which is not a macroscale elastic stress tensor as it does not depend on $\bar{\boldsymbol{u}}$). Since the latter is given, the correct interpretation of the model is still reflected in the formulation (63), i.e. the homogenized behavior of the composite is described by a proper macroscale elastic model, where the additional terms given by the macroscale divergence of σ_L modify the form of the effective body force.

Next we discuss the practical applicability of the model in the context of electroelasticity and magnetoelasticity.

5.2 Applicability to electrosensitive and magnetosensitive composites

Although our formulation is clearly intended to be general with respect to the actual physical system at hand, a highly relevant example resides in the electromagnetic body force arising from the application of an electric and/or magnetic field (A comprehensive, continuum mechanics description of *electroelastic* and *magnetoelastic* materials can be found, for example, in [17, 32]. Elastic composites that are sensitive to electro (magnetic) interactions are relevant in the context of practical applications. Conductive composites, such as carbon fibers reinforced polymer matrices, have received increasing attention both from the experimental (see, e.g. [7]) and the modeling (as in [6, 61]) viewpoints, as they are used in aerospace and automotive industrial applications. Polymeric materials with embedded dielectric inclusions can function as sensors or actuators. They have been experimentally investigated (see, e.g., the recent work [21]) and they can provide a basis for biomimetic applications such as artificial muscles [26]. Composite materials made of ferromagnetic or paramagnetic fibers and particles have been recently theoretical and experimentally investigated (see, e.g., [12, 31], respectively), as the strains that develop as a consequence of an imposed magnetic field can be exploited for biomimetic applications and provide physiological benefits, for example in the context of bone growth (see, e.g., [25]). The electromagnetic force when both an electric and a magnetic field are applied also depends on the velocity of the material and therefore cannot be considered in our formulation. However, whenever electrosensitive or magnetosensitive elastomers (i.e., only responding to the electric and magnetic field, respectively) are considered, the electric and magnetic body forces $\boldsymbol{f}_e$ and $\boldsymbol{f}_m$ read (see, e.g. [34])

$$\boldsymbol{f}_e = q\boldsymbol{E} + (\nabla\boldsymbol{E})^T\boldsymbol{P}, \quad \boldsymbol{f}_m = \boldsymbol{J}\times\boldsymbol{B} + \mu_0^{-1}(\nabla\boldsymbol{B})^T\boldsymbol{M}, \tag{95}$$

where $\boldsymbol{E}$ and $\boldsymbol{B}$ represent the applied electric and magnetic field, μ_0 is the magnetic permeability in vacuum, q and $\boldsymbol{J}$ are free charges and currents, while $\boldsymbol{P}$ and $\boldsymbol{M}$ are the electric polarization and magnetization of the material, which are in general functions of the applied electric and magnetic field and relevant for dielectric and paramagnetic materials, respectively (for ferromagnetic materials the magnetization $\boldsymbol{M}$ is a nonlinear function of the applied magnetic field and can be considered a constant vector in the saturation regime, i.e. for sufficiently high intensity magnetic fields). Both $\boldsymbol{f}_e$ and $\boldsymbol{f}_m$ read as externally applied body forces that are expressed in terms of given quantities. Our model is therefore applicable identifying our body force with either the electric or the magnetic one and assuming that the arising potentials obtained via (1) are locally periodic. The effect of such potentials can be investigated in practice, as they can be obtained controlling the applied electro (magnetic) field as appropriate (see, e.g. [39]). The simplest possible applications of our model resides in the investigation of pure *electrostriction* or *magnetostriction*, i.e. when free charges and currents are not present and the only relevant effect is the elastic strain that develops as a consequence of an applied electric or magnetic field on a dielectric or paramagnetic/ferromagnetic composite, respectively. In this case, assuming a linearized isotropic dielectric/paramagnetic response (which implies that $\boldsymbol{P}$ and $\boldsymbol{M}$ are proportional to $\boldsymbol{E}$ and $\boldsymbol{B}$, respectively) the electric and magnetic forces read

$$\boldsymbol{f}_e = \nabla\phi_e, \quad \phi_e \propto \boldsymbol{E}\cdot\boldsymbol{E}, \quad \boldsymbol{f}_m = \nabla\phi_m, \quad \phi_m \propto \boldsymbol{B}\cdot\boldsymbol{B} \tag{96}$$

i.e. they satisfy our representation with $\boldsymbol{A} = 0$, see (1). The magnetization $\boldsymbol{M}$ is a constant for ferromagnetic materials in the saturation regime, hence we have

$$\boldsymbol{f}_m = \nabla\phi, \quad \phi \propto \boldsymbol{B}\cdot\boldsymbol{M}. \tag{97}$$

Thus, external application of periodic electric or magnetic fields results in body forces given by the periodic scalar potentials defined in (96) and (97). Therefore, our formulation directly applies to the investigation of electrostriction and magnetostriction driven by electric and magnetic fields that are spatially periodic (see, e.g. [53] for a semianalytical and experimental analysis of periodic electric fields in the context of dielectric sensors) and characterized by a wavelength of the same order of the microscale d.

In the next section we discuss the challenges related to the implementation of the model and present a key particular case that retains all the heterogeneities in the potentials at a reduced computational cost.

6 Multiplicative decomposition of the potentials

The effective elastic coefficients and body force that appear in (63) are to be computed by solving (in general, numerically) the six elastic-type cell problems (52-55) as well as the problem (58-61). Although we are assuming macroscopic uniformity of the geometry (see Remark 2), the cell problems still retain a parametric dependence on the macroscale variable $\boldsymbol{x}$, which is encoded in the elastic moduli $\mathbb{C}(\boldsymbol{x},\boldsymbol{y})$ and in the local volume load $\boldsymbol{f}_{\boldsymbol{y}}$. As a consequence, these cell problems are in principle to be solved for each macroscale point belonging to the homogenized domain Ω_H. In fact, the effective problem for composites is usually simplified by assuming macroscopic uniformity of the constituents' elasticity tensors, so that only locally oscillating, or piecewise constant moduli (as in [43, 44]) are considered in practice. The most general assumption that would permit to eliminate the macroscale dependence from the cell problems (52-55) is a multiplicative decomposition of the type

$$\mathbb{C} = \mathbb{C}^M(\boldsymbol{x})\mathbb{C}^L(\boldsymbol{y}), \tag{98}$$

as in this case macroscale variations of the elasticity tensor could be factorized via an ansatz of the type $\boldsymbol{u}^{(1)} = \chi\mathbb{C}^M\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})$. This way, the contribution due to $\mathbb{C}^M(\boldsymbol{x})$ would only appear in the homogenized balance equations (63), whereas the cell problems would be analogous to (52-55) with $\mathbb{C}$ replaced by $\mathbb{C}^L$, and therefore would depend on the microscale $\boldsymbol{y}$ only. However, our model is intrinsically more complex than the standard one, as macroscale heterogeneities in the cell problem (58-61) are also encoded in the local volume load $\boldsymbol{f}_{\boldsymbol{y}}(\boldsymbol{x},\boldsymbol{y})$. Furthermore, macroscale heterogeneities of the elasticity tensor of the type (98) cannot be factorized from the cell problem (58-61).

We now discuss the key particular case of a multiplicative decomposition of the potentials that leads to a complete decoupling between microscale and macroscale computations. We assume that the elasticity tensors $\mathbb{C}^I$ and $\mathbb{C}^{II}$ are $\boldsymbol{x}$-constant, i.e.

$$\mathbb{C}^I = \mathbb{C}^I(\boldsymbol{y}); \quad \mathbb{C}^{II} = \mathbb{C}^{II}(\boldsymbol{y}), \tag{99}$$

and that both the scalar and the vector potential can be multiplicatively decomposed into a locally varying, $\boldsymbol{x}$-constant component, and a $\boldsymbol{y}$-constant, macroscale one. More specifically, we assume that there exist scalar functions $\phi^M(\boldsymbol{x})$, $g^M(\boldsymbol{x})$, $h^M(\boldsymbol{x})$, $\zeta^M(\boldsymbol{x})$ and $\phi^L(\boldsymbol{y})$, $g^L(\boldsymbol{y})$, $h^L(\boldsymbol{y})$, $\zeta^L(\boldsymbol{y})$ satisfying

$$\phi(\boldsymbol{x},\boldsymbol{y}) = \phi^M(\boldsymbol{x})\phi^L(\boldsymbol{y}), \tag{100}$$

and

$$\boldsymbol{A}(\boldsymbol{x},\boldsymbol{y}) = g^M(\boldsymbol{x})g^L(\boldsymbol{y})\boldsymbol{e_1} + h^M(\boldsymbol{x})h^L(\boldsymbol{y})\boldsymbol{e_2} + \zeta^M(\boldsymbol{x})\zeta^L(\boldsymbol{y})\boldsymbol{e_3}. \tag{101}$$

Enforcing the multiplicative decompositions (100) and (101) in the cell problem for the first order displacements (42-45), the local force $\boldsymbol{f_y}$ that appears on the righ hand sides rewrites as

$$\boldsymbol{f_y} = \nabla_{\boldsymbol{y}}\phi + \nabla_{\boldsymbol{y}}\times\boldsymbol{A} = \phi^M\nabla_{\boldsymbol{y}}\phi^L + g^M\nabla_{\boldsymbol{y}}\times\boldsymbol{g}^L + h^M\nabla_{\boldsymbol{y}}\times\boldsymbol{h}^L + \zeta^M\nabla_{\boldsymbol{y}}\times\boldsymbol{\zeta}^L, \tag{102}$$

where the $\boldsymbol{x}$-constant vectors $\boldsymbol{g}^L$, $\boldsymbol{h}^L$, $\boldsymbol{\zeta}^L$ are defined by

$$\boldsymbol{g}^L(\boldsymbol{y}) = g^L(\boldsymbol{y})\boldsymbol{e_1}; \quad \boldsymbol{h}^L(\boldsymbol{y}) = h^L(\boldsymbol{y})\boldsymbol{e_2}; \quad \boldsymbol{\zeta}^L(\boldsymbol{y}) = \zeta^L(\boldsymbol{y})\boldsymbol{e_3}, \tag{103}$$

where $\boldsymbol{e_1}, \boldsymbol{e_2}, \boldsymbol{e_3}$ are the standard unit cartesian vectors that constitute a basis of $\mathbb{R}^3$. The functional form (103) leads to the following ansätze for the first order displacements $\boldsymbol{u}_I^{(1)}$ and $\boldsymbol{u}_{II}^{(1)}$:

$$\boldsymbol{u}_I^{(1)}(\boldsymbol{x},\boldsymbol{y}) = \chi^I(\boldsymbol{y})\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})(\boldsymbol{x}) + \tilde{\boldsymbol{u}}_I(\boldsymbol{x},\boldsymbol{y}), \tag{104}$$

$$\boldsymbol{u}_{II}^{(1)} = \chi^{II}(\boldsymbol{y})\xi_{\boldsymbol{x}}(\bar{\boldsymbol{u}})(\boldsymbol{x}) + \tilde{\boldsymbol{u}}_{II}(\boldsymbol{x},\boldsymbol{y}), \tag{105}$$

where the auxiliary variables $\tilde{\boldsymbol{u}}_I$ and $\tilde{\boldsymbol{u}}_{II}$ can be specified in terms of the newly introduced functions as follows:

$$\tilde{\boldsymbol{u}}^I(\boldsymbol{x},\boldsymbol{y}) = \phi^M(\boldsymbol{x})\tilde{\boldsymbol{u}}_I^{\phi}(\boldsymbol{y}) + g^M(\boldsymbol{x})\tilde{\boldsymbol{u}}_I^{g}(\boldsymbol{y}) + h^M(\boldsymbol{x})\tilde{\boldsymbol{u}}_I^{h}(\boldsymbol{y}) + \zeta^M(\boldsymbol{x})\tilde{\boldsymbol{u}}_I^{\zeta}(\boldsymbol{y}), \tag{106}$$

$$\tilde{\boldsymbol{u}}^{II}(\boldsymbol{x},\boldsymbol{y}) = \phi^M(\boldsymbol{x})\tilde{\boldsymbol{u}}_{II}^{\phi}(\boldsymbol{y}) + g^M(\boldsymbol{x})\tilde{\boldsymbol{u}}_{II}^{g}(\boldsymbol{y}) + h^M(\boldsymbol{x})\tilde{\boldsymbol{u}}_{II}^{h}(\boldsymbol{y}) + \zeta^M(\boldsymbol{x})\tilde{\boldsymbol{u}}_{II}^{\zeta}(\boldsymbol{y}), \tag{107}$$

where the third rank tensors solve again the cell problems (52-55) (which are now $\boldsymbol{y}$-dependent only by means of (99)), while the auxiliary variables $\tilde{\boldsymbol{u}}_I^{\phi}(\boldsymbol{y})$, $\tilde{\boldsymbol{u}}_I^{g}(\boldsymbol{y})$, $\tilde{\boldsymbol{u}}_I^{h}(\boldsymbol{y})$, $\tilde{\boldsymbol{u}}_I^{\zeta}(\boldsymbol{y})$, $\tilde{\boldsymbol{u}}_{II}^{\phi}(\boldsymbol{y})$, $\tilde{\boldsymbol{u}}_{II}^{g}(\boldsymbol{y})$, $\tilde{\boldsymbol{u}}_{II}^{h}(\boldsymbol{y})$, $\tilde{\boldsymbol{u}}_{II}^{\zeta}(\boldsymbol{y})$ solve the following cell problems that do no retain any macroscale dependency:

$$\nabla_{\boldsymbol{y}}\cdot\left(\mathbb{C}^I\xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^{\phi})\right) + \nabla_{\boldsymbol{y}}\phi^L = \boldsymbol{0} \quad \text{in } \Omega_I, \tag{108}$$

$$\nabla_{\boldsymbol{y}}\cdot\left(\mathbb{C}^{II}\xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^{\phi})\right) + \nabla_{\boldsymbol{y}}\phi^L = \boldsymbol{0} \quad \text{in } \Omega_{II}, \tag{109}$$

$$\mathbb{C}^I\xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^{\phi})\boldsymbol{n} - \mathbb{C}^{II}\xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^{\phi})\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{110}$$

$$\tilde{\boldsymbol{u}}_I^{\phi} = \tilde{\boldsymbol{u}}_{II}^{\phi} \quad \text{on } \Gamma, \tag{111}$$

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^g)\right) + \nabla_{\boldsymbol{y}} \times \boldsymbol{g}^L = 0 \quad \text{in } \Omega_I, \tag{112}$$

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^g)\right) + \nabla_{\boldsymbol{y}} \times \boldsymbol{g}^L = 0 \quad \text{in } \Omega_{II}, \tag{113}$$

$$\mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^g)\boldsymbol{n} - \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^g)\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{114}$$

$$\tilde{\boldsymbol{u}}_I^g = \tilde{\boldsymbol{u}}_{II}^g \quad \text{on } \Gamma, \tag{115}$$

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^h)\right) + \nabla_{\boldsymbol{y}} \times \boldsymbol{h}^L = \boldsymbol{0} \quad \text{in } \Omega_I, \tag{116}$$

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^h)\right) + \nabla_{\boldsymbol{y}} \times \boldsymbol{h}^L = \boldsymbol{0} \quad \text{in } \Omega_{II}, \tag{117}$$

$$\mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^h)\boldsymbol{n} - \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^h)\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{118}$$

$$\tilde{\boldsymbol{u}}_I^h = \tilde{\boldsymbol{u}}_{II}^h \quad \text{on } \Gamma, \tag{119}$$

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^\zeta)\right) + \nabla_{\boldsymbol{y}} \times \boldsymbol{\zeta}^L = \boldsymbol{0} \quad \text{in } \Omega_I, \tag{120}$$

$$\nabla_{\boldsymbol{y}} \cdot \left(\mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^\zeta)\right) + \nabla_{\boldsymbol{y}} \times \boldsymbol{\zeta}^L = \boldsymbol{0} \quad \text{in } \Omega_{II}, \tag{121}$$

$$\mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^\zeta)\boldsymbol{n} - \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^\zeta)\boldsymbol{n} = \boldsymbol{0} \quad \text{on } \Gamma, \tag{122}$$

$$\tilde{\boldsymbol{u}}_I^\zeta = \tilde{\boldsymbol{u}}_{II}^\zeta \quad \text{on } \Gamma. \tag{123}$$

Each of the above differential problems is equipped with periodic conditions on the cell boundary and is well-posed (see Remark 3) once a further condition to ensure uniqueness is imposed. We finally substitute the representations for the potentials (101) and the auxiliary quantities (106-107) into (66) to obtain the effective body force, namely

$$\begin{aligned}
&\hat{\boldsymbol{f}}(\boldsymbol{x}) = \left\langle \phi^L \right\rangle \nabla_{\boldsymbol{x}} \phi^M(\boldsymbol{x}) + \left\langle g^L \right\rangle \nabla_{\boldsymbol{x}} \times \boldsymbol{g}^M(\boldsymbol{x}) + \\
&\left\langle h^L \right\rangle \nabla_{\boldsymbol{x}} \times \boldsymbol{h}^M(\boldsymbol{x}) + \left\langle \zeta^L \right\rangle \nabla_{\boldsymbol{x}} \times \boldsymbol{\zeta}^M(\boldsymbol{x}) + \nabla_{\boldsymbol{x}} \cdot \left(\phi^M(\boldsymbol{x}) \left\langle \mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^\phi) \right\rangle_I\right) \\
&+ \nabla_{\boldsymbol{x}} \cdot \left(g^M(\boldsymbol{x}) \left\langle \mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^g) \right\rangle_I\right) + \nabla_{\boldsymbol{x}} \cdot \left(h^M(\boldsymbol{x}) \left\langle \mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^h) \right\rangle_I\right) \\
&+ \nabla_{\boldsymbol{x}} \cdot \left(\zeta^M(\boldsymbol{x}) \left\langle \mathbb{C}^I \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_I^\zeta) \right\rangle_I\right) + \nabla_{\boldsymbol{x}} \cdot \left(\phi^M(\boldsymbol{x}) \left\langle \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^\phi) \right\rangle_{II}\right) \\
&+ \nabla_{\boldsymbol{x}} \cdot \left(g^M(\boldsymbol{x}) \left\langle \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^g) \right\rangle_{II}\right) + \nabla_{\boldsymbol{x}} \cdot \left(h^M(\boldsymbol{x}) \left\langle \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^h) \right\rangle_{II}\right) \\
&+ \nabla_{\boldsymbol{x}} \cdot \left(\zeta^M(\boldsymbol{x}) \left\langle \mathbb{C}^{II} \xi_{\boldsymbol{y}}(\tilde{\boldsymbol{u}}_{II}^\zeta) \right\rangle_{II}\right),
\end{aligned} \tag{124}$$

where the $\boldsymbol{y}$-constant vectors $\boldsymbol{g}^M$, $\boldsymbol{h}^M$, $\boldsymbol{\zeta}^M$ are defined by

$$\boldsymbol{g}^M(\boldsymbol{x}) = g^M(\boldsymbol{x})\boldsymbol{e_1}; \quad \boldsymbol{h}^M(\boldsymbol{x}) = h^M(\boldsymbol{x})\boldsymbol{e_2}; \quad \boldsymbol{\zeta}^M(\boldsymbol{x}) = \zeta^M(\boldsymbol{x})\boldsymbol{e_3}. \tag{125}$$

Although we have introduced a more complicated notation, the resulting model can actually be solved at a greatly reduced computational cost when compared to the general one presented in section 4. In fact, a multiplicative decomposition of the potentials leads to a complete decoupling between macroscale and microscale problems. Given the geometry of the cell Ω and the macroscopically uniform constituents' elastic moduli $\mathbb{C}^{I}$ and $\mathbb{C}^{II}$, it is just sufficient to solve the six cell elastic-type cell problems (52-55) and the four elastic cell problems (108-111), (112-115), (116-119), (120-123) once and for all. The results, in terms of the auxiliary tensors $\mathbb{M}^{I}$, $\mathbb{M}^{II}$ and microscale strains of the auxiliary variables $\tilde{\boldsymbol{u}}_{I}^{\phi}$, $\tilde{\boldsymbol{u}}_{II}^{\phi}$, $\tilde{\boldsymbol{u}}_{I}^{g}$, $\tilde{\boldsymbol{u}}_{II}^{g}$, $\tilde{\boldsymbol{u}}_{I}^{h}$, $\tilde{\boldsymbol{u}}_{II}^{h}$, $\tilde{\boldsymbol{u}}_{I}^{\zeta}$, $\tilde{\boldsymbol{u}}_{II}^{\zeta}$ are then simply to be plugged into the definition (64) and (124) to recover the effective elasticity tensor $\tilde{\mathbb{C}}$ and volume load $\hat{\boldsymbol{f}}$ that are needed to completely specify the homogenized balance equations (63).

The cell problems related to standard asymptotic homogenization of linear elastic composites (52-55) are in general to be solved via numerical simulations in three dimensions, taking care of a sufficient refinement to properly capture the role of the boundary load across the interface between phases. The computational potential of asymptotic homogenization has been recently investigated in [43], where details of the finite elements computational procedure are also discussed. Moreover, whenever aligned fibers are taken into account, it is possible to study the problems in two dimensions, and semi-analytical results can be obtained by means of the theory of analytic functions and complex variables method (see, e.g. [30]) for cylindrical aligned fibers with circular or elliptical base, as done for example in [36, 37, 49, 50]. The cell problems that are arising from our new formulation (108-111), (112-115), (116-119), and (120-123) are linear elastic cell problems as well, and they are in general simpler than the classical ones, as they are equipped with continuity of the auxiliary displacements and stresses, and also when discontinuties of the elastic coefficients occur they are solely driven by fine scale variations of the potentials' components.

Next, we conclude our manuscript by discussing the limitations of the model, open challenges, and further developments of the work.

7 Conclusions

We have derived new balance equations that describe the homogenized behavior of linear elastic composites subject to inhomogeneous body forces given by a Helmholtz decomposition. We have considered the linear elastostatic problem (8-12) as the starting point of our formulation. Next, we have enforced the length scale separation between the inter-inclusion (or fiber) distance (*the microscale*) and the average size of the composite (*the macroscale*) to apply the asymptotic homogenization technique accounting for microscale periodicity of every material property and field, including the scalar and vector potentials that determine the external body force in (1). The results derived in Section 4 show that the system of PDEs (8-12), which are characterized by a locally unbounded body force (cf. Remark 1), is *homogenizable*, as the macroscale model (63) reads as a linear elastic problem where the effective volume load (66) is given in terms of well-defined quantities. The major difference with respect to the classical formulation resides in the specific contributions appearing in (66), which account for microscale variations of the potentials and are to be computed solving the well-posed auxiliary elastic problem (58-61). The results are discussed in Section 5 and compared with previous modeling approaches in Section 5.1.

The new model is relevant to a wide range of physical scenarios, as it holds for a general external force. In Section 5.2 we have emphasized its applicability to electrosensitive and magnetosensitive materials. In this case, it is straightforward to show how the model should be tested by exploiting particular cases where the body force is explicitly given in terms of the potentials, which are in turn directly controlled by application of the electric or magnetic fields. We nonetheless remark that magnetic and electric torques arising from misalignment between the applied magnetic (electric) field and its associated magnetization (polarization), such as those studied for soft magnetic bodies in [1] are not considered in our framework. In fact, these effects (which also pertain homogeneous applied fields) should be modeled accounting for a possibly non-symmetric stress tensor, as done for example in [55], and such an extension would be complementary to our work and represent an interesting addition to the current literature. However, our formulation is in principle applicable to other contexts, such as thermoelasticity. In the latter case, whenever coupling effects can be neglected (see, e.g. [23]), the contribution due to an external heat source reads as a volume load which is linear in the gradient of the temperature. Therefore, interesting applications of our model reside in the investigation of the strains induced by the application of a (periodic) radiation source, which can be relevant for the study of soft tissue imaging, see, e.g. [33].

Our model is also characterized by key limitations and is open to a number of improvements. The current formulation applies to linear elastic composites and should be extended to a more general finite-elasticity framework to address large strains induced by electric and magnetic fields that may be relevant to biomimetic applications, as in [26]. There exist a few examples in the literature dealing with asymptotic homogenization of nonlinear problems (see, e.g., [24] and [13] for examples of applications to electroactive continua and growing poroelastic media, respectively). It would an interesting challenge to generalize our formulation to nonlinear constitutive equations (see, e.g., [15, 16] for nonlinear magnetoelasticity and electroelasticity, respectively). However, the computational complexity that characterizes nonlinear homogenized models (where the macro and micro scales are typically two-way coupled) and efficient model reductions are still a matter of debate (see, e.g., the numerical strategies suggested in [13]). Furthermore, as the body force is given, our formulation does not apply to coupled problems such as those arising in the analysis of electroactive materials where the piezoelectric effect and inverse magnetostriction are to be considered (i.e. when the dependency of the electric and magnetic response on the strains cannot be neglected, see e.g. [38] and the asymptotic homogenization of a fully coupled thermo-magneto-electroelastic problem reported in [56]). In this case, it would be nontrivial to tackle locally unbounded effects driven by external fields, as the contributions that formally read as a body force for the elastic problem would depend on the variables of a fully coupled system of PDEs.

Since we are dealing with a two-scale formal expansions formalism, it is beyond the scope of the work to tackle the problem via a rigorous two-scale convergence approach. However, a challenging and natural theoretical development of this work resides in adapting the strategies depicted, for example, in [11], to prove rigorous convergence theorems in a weak formulation setting.

The next natural step is the application of the model to physically relevant scenarios (such as those depicted in section 5.2) and its (numerical) implementation, which will also enable a proper comparison with previous modeling approaches, such as [8], under a consistent set of assumptions. The particular case depicted in section 6 can serve as a solid starting point, as in this case the arising cell problems are to be solved once and for all, possibly following and adapting the guidelines provided in [43]. Comparison between numerical results and experimental data will allow model validation, and predictions from the model could be used to support the optimal design of active elastomers.

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